**Partial fraction 1**

1. Partial fraction: $\frac{9x-2}{x^{2}-x-6}$

 $\frac{9x-2}{x^{2}-x-6}=\frac{9x-2}{\left(x-3\right)\left(x+2\right)}=\frac{\left[\frac{9x-2}{x+2}\right]\_{x=3}}{x-3}+\frac{\left[\frac{9x-2}{x-3}\right]\_{x=-2}}{x+2}=\frac{\left[\frac{9(3)-2}{3+2}\right]}{x-3}+\frac{\frac{9(-2)-2}{(-2)-3}}{x+2}=\frac{5}{x-3}+\frac{4}{x+2}$

**2.** Partial fraction: $\frac{3x}{x^{3}-1}$

 (1) $x^{3}-1=\left(x-1\right)\left(x^{2}+x+1\right)$

 (2) $\left(x-1\right)^{2}=\left(x^{2}+x+1\right)-3x$

 $3x=\left(x^{2}+x+1\right)-\left(x-1\right)^{2}$

 (3) $\frac{3x}{x^{3}-1}=\frac{\left(x^{2}+x+1\right)-\left(x-1\right)^{2}}{\left(x-1\right)\left(x^{2}+x+1\right)}=\frac{1}{x-1}-\frac{x-1}{x^{2}+x+1}$

3. Partial fraction: $\frac{x^{2}+1}{\left(x-1\right)^{2}\left(x^{2}+2x+2\right)}$

 Let $\frac{x^{2}+1}{\left(x-1\right)^{2}\left(x^{2}+2x+2\right)}=\frac{A}{\left(x-1\right)^{2}}+\frac{f(x)}{\left(x-1\right)\left(x^{2}+2x+2\right)}$

 Then $A=\left[\frac{x^{2}+1}{x^{2}+2x+2}\right]\_{x=1}=\frac{1^{2}+1}{1^{2}+2(1)+2}=\frac{2}{5}$

 Hence $\frac{x^{2}+1}{\left(x-1\right)^{2}\left(x^{2}+2x+2\right)}=\frac{\frac{2}{5}}{\left(x-1\right)^{2}}+\frac{f(x)}{\left(x-1\right)\left(x^{2}+2x+2\right)}$

 $∴\left(x-1\right)f\left(x\right)=\left(x^{2}+1\right)-\frac{2}{5}\left(x^{2}+2x+2\right)=\frac{1}{5}\left(x-1\right)(3x-1)$

 $∴f\left(x\right)=\frac{1}{5}\left(3x-1\right)$

 Hence $\frac{x^{2}+1}{\left(x-1\right)^{2}\left(x^{2}+2x+2\right)}=\frac{\frac{2}{5}}{\left(x-1\right)^{2}}+\frac{1}{5}\frac{\left(3x-1\right)}{\left(x-1\right)\left(x^{2}+2x+2\right)}…\left(1\right)$

 Consider $\frac{3x-1}{\left(x-1\right)\left(x^{2}+2x+2\right)}=\frac{B}{x-1}+\frac{g(x)}{x^{2}+2x+2}$

 Then $B=\left[\frac{3x-1}{x^{2}+2x+2}\right]\_{x=1}=\frac{3-1}{1+2+2}=\frac{2}{5}$

 $\frac{3x-1}{\left(x-1\right)\left(x^{2}+2x+2\right)}=\frac{\frac{2}{5}}{x-1}+\frac{g(x)}{x^{2}+2x+2}$

 $(x-1)g\left(x\right)=3x-1-\frac{2}{5}\left(x^{2}+2x+2\right)$

 $g\left(x\right)=-\frac{1}{5}(2x-9)$

 $\frac{3x-1}{\left(x-1\right)\left(x^{2}+2x+2\right)}=\frac{\frac{2}{5}}{x-1}+\frac{-\frac{1}{5}(2x-9)}{x^{2}+2x+2}…(2)$

 $(2)\downright \left(1\right), \frac{x^{2}+1}{\left(x-1\right)^{2}\left(x^{2}+2x+2\right)}=\frac{\frac{2}{5}}{\left(x-1\right)^{2}}+\frac{\frac{2}{25}}{x-1}+\frac{-\frac{1}{25}(2x-9)}{x^{2}+2x+2}$

4. Partial fraction: $\frac{5x^{2}+x+6}{ \left(3-2x\right)\left(x^{2}+4\right)}$

 Let $\frac{5x^{2}+x+6}{ \left(3-2x\right)\left(x^{2}+4\right)}=\frac{A}{3-2x}+\frac{f(x)}{x^{2}+4}$

 Then $A=\left[\frac{5x^{2}+x+6}{x^{2}+4}\right]\_{x=\frac{3}{2}}=\frac{5\left(\frac{3}{2}\right)^{2}+\left(\frac{3}{2}\right)+6}{\left(\frac{3}{2}\right)^{2}+4}=3$

 Hence $\frac{5x^{2}+x+6}{ \left(3-2x\right)\left(x^{2}+4\right)}=\frac{3}{3-2x}+\frac{f(x)}{x^{2}+4}=\frac{3\left(x^{2}+4\right)+\left(3-2x\right)f(x)}{\left(3-2x\right)\left(x^{2}+4\right)}$

 $f\left(x\right)=\frac{5x^{2}+x+6-3\left(x^{2}+4\right)}{3-2x}=\frac{2x^{2}+x-6}{3-2x}=\frac{\left(2x-3\right)(x+2)}{3-2x}=-(x+2)$

 $∴\frac{5x^{2}+x+6}{ \left(3-2x\right)\left(x^{2}+4\right)}=\frac{3}{3-2x}-\frac{x+2}{x^{2}+4}$

5. Partial fraction: $\frac{x+1}{\left(x^{2}+1\right)\left(x^{2}+x+1\right)}$

The two factors in the denominator are quite close.

 However, the numerator $x+1$ cannot be rewriten as $\left(x^{2}+x+1\right)-\left(x^{2}+1\right)$.

 Now, $\left(x-1\right)\left(x^{2}+x+1\right)=x^{3}-1$ is familiar.

 We don’t like the $x^{3}$ term.

 Working on another factor, $x\left(x^{2}+1\right)=x^{3}+x$

 Hence $x\left(x^{2}+1\right)-\left(x-1\right)\left(x^{2}+x+1\right)=x+1$. Bingo!

 $∴\frac{x+1}{\left(x^{2}+1\right)\left(x^{2}+x+1\right)}=\frac{x\left(x^{2}+1\right)-\left(x-1\right)\left(x^{2}+x+1\right)}{\left(x^{2}+1\right)\left(x^{2}+x+1\right)}=\frac{x}{x^{2}+x+1}-\frac{x-1}{x^{2}+x}$

6. Partial fractions $\frac{7x^{2}-10x+10}{\left(x-1\right)^{3}}$

 $\frac{7x^{2}-10x+10}{\left(x-1\right)^{3}}=\frac{7\left[\left(x-1\right)+1\right]^{2}-10\left[\left(x-1\right)+1\right]+10}{\left(x-1\right)^{3}}=\frac{7\left[\left(x-1\right)^{2}+2\left(x-1\right)+1\right]-10\left[\left(x-1\right)+1\right]+10}{\left(x-1\right)^{3}}$

 $=\frac{7\left(x-1\right)^{2}+4\left(x-1\right)+7}{\left(x-1\right)^{3}}=\frac{7}{x-1}+\frac{4}{\left(x-1\right)^{2}}+\frac{7}{\left(x-1\right)^{3}}$

7. Partial fraction: $ \frac{9x^{2}+34x+14}{\left(x+2\right)\left(x^{2}-x-12\right)}$

 $\frac{9x^{2}+34x+14}{\left(x+2\right)\left(x^{2}-x-12\right)}=\frac{9x^{2}+34x+14}{\left(x+2\right)\left(x+3\right)\left(x-4\right)}$

 $=\frac{\left[\frac{9x^{2}+34x+14}{\left(x+3\right)\left(x-4\right)}\right]\_{x=-2}}{x+2}+\frac{\left[\frac{9x^{2}+34x+14}{\left(x+2\right)\left(x-4\right)}\right]\_{x=-3}}{x+3}+\frac{\left[\frac{9x^{2}+34x+14}{\left(x+2\right)\left(x+3\right)}\right]\_{x=4}}{x-4}$

 $=\frac{3}{x+2}-\frac{1}{x+3}+\frac{7}{x-4}$

8. Partial fraction: $\frac{x-1}{\left(3x-1\right)\left(2x+5\right)}$

 $\frac{x-1}{\left(3x-1\right)\left(2x+5\right)}=\frac{\left[\frac{x-1}{2x+5}\right]\_{x=\frac{1}{3}}}{3x-1}+\frac{\left[\frac{x-1}{3x-1}\right]\_{x=-\frac{5}{2}}}{2x+5}=\frac{\left[\frac{\frac{1}{3}-1}{2\left(\frac{1}{3}\right)+5}\right]}{3x-1}+\frac{\left[\frac{-\frac{5}{2}-1}{3\left(-\frac{5}{2}\right)-1}\right]}{2x+5}=\frac{- \frac{2}{17}}{3x-1}+\frac{\frac{7}{17}}{2x+5}$

9. Partial fraction: $\frac{x^{4}}{\left(x^{2}-x+1\right)\left(x^{2}+2\right)^{2} }$

 (1) $\left(x^{2}+x+1\right)\left(x^{2}-x+1\right)=\left(x^{2}+1\right)^{2}-x^{2}=x^{4}+x^{2}+1$

 $\left(x^{2}+2\right)^{2}=x^{4}+4x^{2}+4$

 (2) Let $A\left(x^{4}+x^{2}+1\right)+B\left(x^{4}+4x^{2}+4\right)=x^{4}⟹\left\{\begin{array}{c}A+B=1\\A+4B=0\end{array}\right.⟹A=\frac{4}{3} ,B=-\frac{1}{3}$

 (3) $\frac{x^{4}}{\left(x^{2}-x+1\right)\left(x^{2}+2\right)^{2} }=\frac{\frac{4}{3}\left(x^{2}+x+1\right)\left(x^{2}-x+1\right)-\frac{1}{3}\left(x^{2}+2\right)^{2}}{\left(x^{2}-x+1\right)\left(x^{2}+2\right)^{2} }=\frac{4}{3}\left[\frac{x^{2}+x+1}{\left(x^{2}+2\right)^{2}}\right]-\frac{1}{3}\left[\frac{1}{x^{2}-x+1}\right]$

 $=\frac{4}{3}\left[\frac{\left(x^{2}+2\right)+\left(x-1\right)}{\left(x^{2}+2\right)^{2}}\right]-\frac{1}{3}\left[\frac{1}{x^{2}-x+1}\right]=\frac{4}{3}\left[\frac{1}{x^{2}+2}\right]+\frac{4}{3}\left[\frac{x-1}{\left(x^{2}+2\right)^{2}}\right]-\frac{1}{3}\left[\frac{1}{x^{2}-x+1}\right]$

10. Partial fraction: $\frac{3x-2}{(x+1)^{2}\left(x-3\right)^{2}}$

 **(1)** $\frac{3x-2}{(x+1)\left(x-3\right)}=\frac{\left[\frac{3x-2}{x-3}\right]\_{x=-1}}{x+1}+\frac{\left[\frac{3x-2}{x+1}\right]\_{x=3}}{x-3}=\frac{5}{4} \frac{1}{x+1}+\frac{7}{4} \frac{1}{x-3}…(1)$

 **(2)** $\frac{1}{(x+1)\left(x-3\right)}=\frac{\left[\frac{1}{x-3}\right]\_{x=-1}}{x+1}+\frac{\left[\frac{1}{x+1}\right]\_{x=3}}{x-3}=-\frac{1}{4} \frac{1}{x+1}+\frac{1}{4} \frac{1}{x-3}…(2)$

 **(3)** $\left(1\right)×\left(2\right), \frac{3x-2}{(x+1)^{2}\left(x-3\right)^{2}}=-\frac{5}{16}\frac{1}{(x+1)^{2}}-\frac{1}{8}\frac{1}{(x+1)\left(x-3\right)}+\frac{7}{16}\frac{1}{\left(x-3\right)^{2}}$

 $=-\frac{5}{16}\frac{1}{(x+1)^{2}}-\frac{1}{8}\left(-\frac{1}{4} \frac{1}{x+1}+\frac{1}{4} \frac{1}{x-3}\right)+\frac{7}{16}\frac{1}{\left(x-3\right)^{2}}$ , by (2)

 $=-\frac{5}{16}\frac{1}{(x+1)^{2}}+\frac{1}{32} \frac{1}{x+1}-\frac{1}{32} \frac{1}{x-3}+\frac{7}{16}\frac{1}{\left(x-3\right)^{2}}$

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